A Neat Vector Expression for The Direction of Refraction

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June 10, 2018

1 Introduction

Snell's law is

$$n_1 \sin \theta_2 = n_2 \sin \theta_2$$

where n_1, n_2 are the refractive indices of the two materials. The angles θ_1, θ_2 are the angles that the incoming and refracted rays make with the normal direction. One could calculate $\theta_2 = \arcsin\left(\frac{n_1}{n_2} \sin \theta_1\right)$ and then calculate the direction of refraction from θ_2 using sine and cosine and a bit of vector arithmetic. All told, that's a very expensive way of calculating the direction of refraction. In this document, we derive an economical vector expression instead.

2 Vector expression

Suppose that the direction of the incoming ray is v_1, and the outgoing ray is v_2. We make no assumption about the length of v_1, and pose no requirement on the length of v_2.

Snell's can be translated as

$$n_1 \frac{\langle v_1, t \rangle}{|v_1|} = n_2 \frac{\langle v_2, t \rangle}{|v_2|}$$

where t is the transverse vector, $t = \frac{v_i - \langle v_1, n \rangle n}{|v_i - \langle v_1, n \rangle n|}$ and n is the normal vector. We will assume $|n| = 1$.

Now we find the components of v_2 with respect to n and t

$$\langle v_2, t \rangle = \frac{n_1}{n_2} \frac{|v_2|}{|v_1|} \langle v_1, t \rangle$$

$$\langle v_2, n \rangle = -\sqrt{|v_2|^2 - \langle v_2, t \rangle^2} = -\sqrt{|v_2|^2 - \frac{n_1^2}{n_2^2} \frac{|v_2|^2}{|v_1|^2} \langle v_1, t \rangle^2}$$

Note that we've chosen the sign of $\langle v_2, n \rangle$ to be negative. This choice is arbitrary, and Snell's law itself does not make it for us. Snell's law only says that the signs of $\langle v_1, n \rangle$ and $\langle v_2, n \rangle$ are equal.

Now we can calculate $v_2 = \langle v_2, t \rangle t + \langle v_2, n \rangle n$. If we substitute everything in, we get this somewhat messy expression

$$v_2 = \frac{n_1}{n_2} \frac{|v_2|}{|v_1|} \langle v_1, t \rangle t - \sqrt{|v_2|^2 - \frac{n_1^2}{n_2^2} \frac{|v_2|^2}{|v_1|^2} \langle v_1, t \rangle^2} n$$

Note that when we wrote down $v_2 = \langle v_2, t \rangle t + \langle v_2, n \rangle n$, we implicitly decided that $|v_2| = |v_1|$. Thus we have

$$v_2 = \frac{n_1}{n_2} \langle v_1, t \rangle t - \sqrt{|v_1|^2 - \left(\frac{n_1}{n_2} \right)^2 \langle v_1, t \rangle^2} n$$

Now remember that $t = \frac{\tau}{|\tau|}$ where $\tau = v_1 - \langle v_1, n \rangle n$. We can get rid of t in favor of τ, and clean up the expression further. Note that $\langle v_1, t \rangle = \frac{\langle v_1, \tau \rangle}{|\tau|} = \frac{|v_1|^2 - \langle v_1, n \rangle^2}{|\tau|} = \frac{\langle v_1, t \rangle^2}{|\tau|}$, which implies $\langle v_1, t \rangle = |\tau|$. Using this, we get

$$v_2 = \frac{n_1}{n_2} \tau - \sqrt{|v_1|^2 - \left(\frac{n_1}{n_2} \right)^2 |\tau|^2} n$$

Now define $\omega = \frac{n_1}{n_2} \tau = \frac{n_1}{n_2} (v_1 - \langle v_1, n \rangle n)$ to obtain our preferred expression

$$\boxed{v_2 = \omega - \sqrt{|v_1|^2 - |\omega|^2} n}$$

We implicitly enforced $|v_1| = |v_2|$, but it can be verified again using the above expression. It's easy to check that the condition of total internal refraction is exactly when $|v_1|^2 < |\omega|^2$.