

Exercise 29 (page 105)

To show *existence*, define $\lambda = \int \omega = \int_0^1 f dx$, and $g(x) = \int_0^x f dx - \lambda x$. By Theorem 4-7, we calculate $dg = Dg \cdot dx$. $Dg = f - \lambda$, so $dg = \omega - \lambda dx$ follows. Also note that $g(0) = \int_0^0 f dx - \lambda \cdot 0 = 0$, and $g(1) = \int_0^1 f dx - \lambda = \int_0^1 f dx - \int_0^1 f dx = 0$, so $g(0) = g(1) = 0$.

To show *uniqueness*, suppose that $\omega - \lambda dx = dg$, where $g(0) = g(1)$. Integrating, we get

$$\int_0^1 \omega - \lambda = \int_0^1 dg$$

By Stoke's theorem, we have $\int_0^1 dg = g(1) - g(0) = 0$, so $\lambda = \int_0^1 \omega = \int_0^1 f dx$. So the λ is completely determined by f , and is thus *unique*.

Exercise 34 (page 108)

(a) Following the definitions, we have

$$\partial C_{F,G} = -(C_{F,G})_{(1,0)} + (C_{F,G})_{(1,1)} + (C_{F,G})_{(2,0)} - (C_{F,G})_{(2,1)} - (C_{F,G})_{(3,0)} + (C_{F,G})_{(3,1)}$$

where

$$(C_{F,G})_{(1,0)}(u, v) = C_{F,G}(0, u, v) = F(0, u) - G(0, v) = C_{F_0, G_0}(u, v)$$

$$(C_{F,G})_{(1,1)}(u, v) = C_{F,G}(1, u, v) = F(1, u) - G(1, v) = C_{F_1, G_1}(u, v)$$

$$(C_{F,G})_{(2,0)}(s, v) = C_{F,G}(s, 0, v) = F(s, 0) - G(s, v)$$

$$(C_{F,G})_{(2,1)}(s, v) = C_{F,G}(s, 1, v) = F(s, 1) - G(s, v)$$

$$(C_{F,G})_{(3,0)}(s, u) = C_{F,G}(s, u, 0) = F(s, u) - G(s, 0)$$

$$(C_{F,G})_{(3,1)}(s, v) = C_{F,G}(s, u, 1) = F(s, u) - G(s, 1)$$

Since each F_s is closed, i.e. $F(s, 0) = F(s, 1)$, for all s , we see that $(C_{F,G})_{(2,0)} = (C_{F,G})_{(2,1)}$.

Similarly, since each G_s is closed, $(C_{F,G})_{(3,0)} = (C_{F,G})_{(3,1)}$. Thus, the only surviving terms of $\partial C_{F,G}$ are those due to $(C_{F,G})_{(1,0)}$ and $(C_{F,G})_{(1,1)}$. So $\partial C_{F,G} = C_{F_1, G_1} - C_{F_0, G_0}$ follows.

(b) If $d\omega = 0$ then by (a) and Theorem 4-13 (Stoke's theorem), we get

$$0 = \int_{C_{F,G}} d\omega = \int_{\partial C_{F,G}} \omega = \int_{C_{F_1, G_1}} \omega - \int_{C_{F_0, G_0}} \omega$$

so we conclude $\int_{C_{F_1, G_1}} \omega = \int_{C_{F_0, G_0}} \omega$.