1 - Algebraic Preliminaries

Exercise 4 (*page 85*)

Since $\dim(\Lambda^n(\mathbb{R})) = 1$, and since $\omega \in \Lambda^n(V)$ implies that $f^*\omega \in \Lambda^n(\mathbb{R}^n)$, we know that $f^*\omega = c \det$ for some $c \in \mathbb{R}$. To find out c, we evaluate this equation at $(e_1, \ldots, e_n)$:

$$f^*\omega(e_1, \ldots, e_n) = c \det \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix} = c$$

or

$$\omega(f(e_1), \ldots, f(e_n)) = c$$

Applying Theorem 4-6, we get

$$det(a_{ij}) \cdot \omega(v_1, \ldots, v_n) = c$$

where $\{v_i\}_{i=1}^n$ is a basis for v and the a_{ij} are defined by $f(e_i) = \sum_{j=1}^n a_{ij} v_j$. By hypothesis, $[f(e_1), \ldots, f(e_n)] = \mu$ so that $\det(a_{ij}) = 1$. Also, since ω is the volume element w.r.t. μ and T, we get $\omega(v_1, \ldots, v_n) = 1$, so we conclude $c = 1$. Thus $f^*\omega = \det$ follows.

4 - The Fundamental Theorem of Calculus

Exercise 29 (*page 105*)

To show *existence*, define $\lambda = \int \omega = \int_0^1 f dx$, and $g(x) = \int_0^x f dx - \lambda x$. By Theorem 4-7, we calculate $dg = Dg \cdot dx$. $Dg = f - \lambda$, so $dg = \omega - \lambda dx$ follows. Also note that $g(0) = \int_0^0 f dx - \lambda \cdot 0 = 0$, and $g(1) = \int_0^1 f dx - \lambda = \int_0^1 f dx - \int_0^1 f dx = 0$, so $g(0) = g(1) = 0$.

To show *uniqueness*, suppose that $\omega - \lambda dx = dg$, where $g(0) = g(1)$. Integrating, we get

$$\int_0^1 \omega - \lambda = \int_0^1 dg$$

By Stoke's theorem, we have $\int_0^1 dg = g(1) - g(0) = 0$, so $\lambda = \int_0^1 \omega = \int_0^1 f dx$. So the λ is completely determined by f, and is thus *unique*.

 ISBN10: 0-8053-9021-9

Exercise 34 (*page 108*)

(a) Following the definitions, we have

$$\partial C_{F,G} = -(C_{F,G})_{(1,0)} + (C_{F,G})_{(1,1)} + (C_{F,G})_{(2,0)} - (C_{F,G})_{(2,1)} - (C_{F,G})_{(3,0)} + (C_{F,G})_{(3,1)}$$

where

$$\begin{aligned}
(C_{F,G})_{(1,0)}(u,v) &= C_{F,G}(0,u,v) = F(0,u) - G(0,v) = C_{F_0,G_0}(u,v)\\
(C_{F,G})_{(1,1)}(u,v) &= C_{F,G}(1,u,v) = F(1,u) - G(1,v) = C_{F_1,G_1}(u,v)\\
(C_{F,G})_{(2,0)}(s,v) &= C_{F,G}(s,0,v) = F(s,0) - G(s,v)\\
(C_{F,G})_{(2,1)}(s,v) &= C_{F,G}(s,1,v) = F(s,1) - G(s,v)\\
(C_{F,G})_{(3,0)}(s,u) &= C_{F,G}(s,u,0) = F(s,u) - G(s,0)\\
(C_{F,G})_{(3,1)}(s,v) &= C_{F,G}(s,u,1) = F(s,u) - G(s,1)
\end{aligned}$$

Since each F_s is closed, i.e. $F(s,0) = F(s,1)$, for all s, we see that $(C_{F,G})_{(2,0)} = (C_{F,G})_{(2,1)}$.

Similarly, since each G_s is closed, $(C_{F,G})_{(3,0)} = (C_{F,G})_{(3,1)}$. Thus, the only surviving terms of $\partial C_{F,G}$ are those due to $(C_{F,G})_{(1,0)}$ and $(C_{F,G})_{(1,1)}$. So $\partial C_{F,G} = C_{F_1,G_1} - C_{F_0,G_0}$ follows.

(b) If $d\omega = 0$ then by (a) and Theorem 4-13 (Stoke's theorem), we get

$$0 = \int_{C_{F,G}} d\omega = \int_{\partial C_{F,G}} \omega = \int_{C_{F_1,G_1}} \omega - \int_{C_{F_0,G_0}} \omega$$

so we conclude $\int_{C_{F_1,G_1}} \omega = \int_{C_{F_0,G_0}} \omega$.

 ISBN10: 0-8053-9021-9