Exercise 7 (*page 26*)

Since V and V' are isomorphic (they are both n-dimensional), their counts of m-dimensional substaces are equal. The mapping $\mathfrak{m} \to \mathfrak{m}^0$ together with theorems 1 and 2 then fashions a bijection of the m-dimensional subspaces of $\mathfrak{V}$ with the $n-m$-dimensional subspaces of $\mathfrak{V}'$. **Injectivity:** suppose $\mathfrak{m}_1^0 = \mathfrak{m}_2^0$. Then by theorem 2, we get $\mathfrak{m}_1 = \mathfrak{m}_2$. **Surjectivity:** let $\mathfrak{m}'$ be a subspace of $\mathfrak{V}'$. Then $((\mathfrak{m}')^0)^0 = \mathfrak{m}'$, again by theorem 2.

Exercise 8 (*page 27*)

(c) We need to show $(\mathfrak{m}+\mathfrak{n})^\circ = \mathfrak{m}^\circ \cap \mathfrak{n}^\circ$ and $(\mathfrak{m}\cap\mathfrak{n})^\circ = \mathfrak{m}^\circ + \mathfrak{n}^\circ$. Note that

$$\mathfrak{m}\cap\mathfrak{n} \subset \mathfrak{m}, \mathfrak{n}$$

$$(\mathfrak{m}\cap\mathfrak{n})^\circ \supset \mathfrak{m}^\circ, \mathfrak{n}^\circ$$

Thus

$$(\mathfrak{m}\cap\mathfrak{n})^\circ \supset \mathfrak{m}^\circ + \mathfrak{n}^\circ \tag{1}$$

But then it follows (by applying $\cdot^\circ$ to all sets) that

$$(\mathfrak{m}^\circ\cap\mathfrak{n}^\circ)^\circ \supset (\mathfrak{m}^\circ)^\circ + (\mathfrak{n}^\circ)^\circ$$

$$\mathfrak{m}^\circ\cap\mathfrak{n}^\circ \supset \mathfrak{m} + \mathfrak{n} \tag{2}$$

From (1) also follows that

$$\dim(\mathfrak{m}\cap\mathfrak{n})^\circ \geq \dim(\mathfrak{m}^\circ + \mathfrak{n}^\circ)$$

Letting $N = \dim(\mathfrak{V}), m = \dim\mathfrak{m}, n = \dim(\mathfrak{n})$, we have

$$N - \dim(\mathfrak{m}\cap\mathfrak{n}) \geq \dim(\mathfrak{m}^\circ) + \dim(\mathfrak{n}^\circ) - \dim(\mathfrak{m}^\circ\cap\mathfrak{n}^\circ)$$

$$N - (m + n - \dim(\mathfrak{m}+\mathfrak{n})) \geq N - m + N - n - \dim(\mathfrak{m}^\circ\cap\mathfrak{n}^\circ)$$

$$\dim(\mathfrak{m}+\mathfrak{n})) \geq N - \dim(\mathfrak{m}^\circ\cap\mathfrak{n}^\circ)$$

$$\dim(\mathfrak{m}^\circ\cap\mathfrak{n}^\circ) \geq N - \dim(\mathfrak{m}+\mathfrak{n})) \tag{3}$$

Now, (2) and (3) together imply that

$$\mathfrak{m}^\circ\cap\mathfrak{n}^\circ = (\mathfrak{m}+\mathfrak{n})^\circ$$

and in a similar way as (2) follows from (1), we may turn this formula into $(\mathfrak{m}\cap\mathfrak{n})^\circ = \mathfrak{m}^\circ + \mathfrak{n}^\circ)$.