7 - Bases

Exercise 3 (*page 12*)

Yes, it is true. Suppose $x+y$, $y+z$ and $x+z$ were not linearly independent. For instance, suppose $\exists\alpha,\beta$ s.t. $\alpha(x+y)+\beta(y+z)=x+z$. Then we'd have

$$\alpha(x+y)+\beta(y+z)=x+z$$

$$(1-\alpha)x+(1-\beta)z=y(\alpha+\beta)$$

If any two of $1-\alpha$, $1-\beta$ and $\alpha+\beta$ are zero, then the third one is not zero:

$$1-\alpha=0 \text{ and } 1-\beta=0 \Rightarrow \alpha+\beta=2$$

$$1-\alpha=0 \text{ and } \alpha+\beta=0 \Rightarrow 1-\beta=2$$

$$1-\beta=0 \text{ and } \alpha+\beta=0 \Rightarrow 1-\alpha=2$$

Therefore $\{x,y,z\}$ are linearly dependent. ∎

12 - Dimension of a subspace

Exercise 6 (*page 19*)

By theorem 2, we can find a basisi of $\mathfrak{V}$ s.t. $\{x_1,\cdots,x_m,y_1,\cdots,y_{N-m}\}$, where $N=\dim(\mathfrak{V})$, $\{x_1,\cdots,x_m\}$ is a basis for $\mathfrak{m}$ and $m\neq 0,N$. It is clear that $\mathfrak{n}=\text{span}y_i$ is a complement to $\mathfrak{m}$. Now let $y=a_1x_1+\cdots+a_mx_m+b_1y_1+\cdots+b_{N-m}x_{N-m}$ where not all a_i and not all b_i are zero. This means that $y\notin\mathfrak{m}$ and $y\notin\mathfrak{n}$. Suppose, without loss of generality, that $b_1\neq 0$. Then $\{x_1,\cdots,x_m,y,y_2,\cdots,y_{N-m}\}$ also spans $\mathfrak{V}$, and so $\{y,y_2,\cdots,y_{N-m}\}$ is a different complement to $\mathfrak{m}$. Thus, the answer to **(a)** is "no". As for **(b)**, the only thing to note is that if $\{x_1,\cdots,x_m\}$ is a basis for $\mathfrak{m}$ and $\{y_1,\cdots,y_n\}$ is a basis for $\mathfrak{n}$, and if $\mathfrak{m}$ and $\mathfrak{n}$ are complements, then clearly $\{x_1,\cdots,x_m,y_1,\cdots,y_n\}$ spans $\mathfrak{V}$, and furthermore this set is linearly independent since the $\{x_i\}$ and $\{y_i\}$ are, and since $\mathfrak{m}\cap\mathfrak{n}=\mathfrak{o}$. Thus $\dim\mathfrak{V}=m+n$. ∎

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17 - Annihilators

Exercise 7 (*page 26*)

Since V and V' are isomorphic (they are both n-dimensional), their counts of m-dimensional substaces are equal. The mapping $\mathfrak{m} \to \mathfrak{m}^0$ together with theorems 1 and 2 then fashions a bijection of the m-dimensional subspaces of $\mathfrak{V}$ with the $n-m$-dimensional subspaces of $\mathfrak{V}'$. **Injectivity:** suppose $\mathfrak{m}_1^0 = \mathfrak{m}_2^0$. Then by theorem 2, we get $\mathfrak{m}_1 = \mathfrak{m}_2$. **Surjectivity:** let $\mathfrak{m}'$ be a subspace of $\mathfrak{V}'$. Then $((\mathfrak{m}')^0)^0 = \mathfrak{m}'$, again by theorem 2.

Exercise 8 (*page 27*)

(c) We need to show $(\mathfrak{m}+\mathfrak{n})^\circ = \mathfrak{m}^\circ \cap \mathfrak{n}^\circ$ and $(\mathfrak{m}\cap\mathfrak{n})^\circ = \mathfrak{m}^\circ + \mathfrak{n}^\circ$. Note that

$$\mathfrak{m}\cap\mathfrak{n} \subset \mathfrak{m},\mathfrak{n}$$
$$(\mathfrak{m}\cap\mathfrak{n})^\circ \supset \mathfrak{m}^\circ,\mathfrak{n}^\circ$$

Thus

$$(\mathfrak{m}\cap\mathfrak{n})^\circ \supset \mathfrak{m}^\circ + \mathfrak{n}^\circ \tag{1}$$

But then it follows (by applying $\cdot^\circ$ to all sets) that

$$(\mathfrak{m}^\circ\cap\mathfrak{n}^\circ)^\circ \supset (\mathfrak{m}^\circ)^\circ + (\mathfrak{n}^\circ)^\circ$$
$$\mathfrak{m}^\circ\cap\mathfrak{n}^\circ \supset \mathfrak{m}+\mathfrak{n} \tag{2}$$

From (1) also follows that

$$\dim(\mathfrak{m}\cap\mathfrak{n})^\circ \geq \dim(\mathfrak{m}^\circ + \mathfrak{n}^\circ)$$

Letting $N = \dim(\mathfrak{V}), m = \dim\mathfrak{m}, n = \dim(\mathfrak{n})$, we have

$$N - \dim(\mathfrak{m}\cap\mathfrak{n}) \geq \dim(\mathfrak{m}^\circ) + \dim(\mathfrak{n}^\circ) - \dim(\mathfrak{m}^\circ\cap\mathfrak{n}^\circ)$$
$$N - (m + n - \dim(\mathfrak{m}+\mathfrak{n})) \geq N - m + N - n - \dim(\mathfrak{m}^\circ\cap\mathfrak{n}^\circ)$$
$$\dim(\mathfrak{m}+\mathfrak{n})) \geq N - \dim(\mathfrak{m}^\circ\cap\mathfrak{n}^\circ)$$
$$\dim(\mathfrak{m}^\circ\cap\mathfrak{n}^\circ) \geq N - \dim(\mathfrak{m}+\mathfrak{n})) \tag{3}$$

Now, (2) and (3) together imply that

$$\mathfrak{m}^\circ\cap\mathfrak{n}^\circ = (\mathfrak{m}+\mathfrak{n})^\circ$$

and in a similar way as (2) follows from (1), we may turn this formula into $(\mathfrak{m}\cap\mathfrak{n})^\circ = \mathfrak{m}^\circ + \mathfrak{n}^\circ)$.

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