Exercise 2 (*page 46*)

We will show that if there is $a \in G$ of finite order, then $\cap H = \{e\}$, where H ranges over the subgroups of G such that $H \neq \{e\}$ (i.e. H is non-trivial).

Clearly $\langle a \rangle \supset \cap H$. We now show that $\cap_{H \subset \langle a \rangle} H = \{e\}$. Pick any $H \subset \langle a \rangle$. There exists a smallest $n > 0$ s.t. $a^n \in H$. Suppose $n \nmid m$ for some m. Then $x = xn + y$ for some $0 < y < n$. Thus $a^m = a^{xn+y} \Rightarrow a^{m-xn} = a^y \in G$ which contradicts minimality of n.

We can now easily state that $a^k \in G$ iff $n|k$. Thus $a^k \in \cap_{H \subset \langle a \rangle} H$ iff $p|k$ for all $p \in \mathbb{Z}$. This implies that $k = 0$, so $\cap_{H \subset \langle a \rangle} H = \{e\}$.

 ISBN10: 0-471-01090-1