

1 - Norm and inner product

Exercise 7 (page 4)

(a) Suppose T preserves inner product. Then $|Tx| = \sqrt{\langle Tx, Tx \rangle} = \sqrt{\langle x, x \rangle} = |x|$. Thus T is norm preserving. Now suppose T preserves norm. Then by theorem 1-2(5), we have $\langle Tx, Ty \rangle = \frac{|Tx+Ty|^2 - |Tx-Ty|^2}{4} = \frac{|T(x+y)|^2 - |T(x-y)|^2}{4} = \frac{|x+y|^2 - |x-y|^2}{4} = \langle x, y \rangle$, and so T preserves inner product. (b) If T was not 1-to-1, then there would exist x, y with $x \neq y$ (i.e. $|x - y| > 0$), yet with $Tx = Ty$ (i.e. $|Tx - Ty| = 0$). But by hypothesis $|Tx - Ty| = |x - y| = 0$, so T must have an inverse, which is clearly also norm preserving (and therefore it also preserves the inner product).