2 - Continuity

Exercise 18 (*page 36*)

Let $U \subset Y$ be an open set, and let $p \in f^{-1}(U)$. There is an m such that $p \in A_m$. Note that $(f|_{A_{m+1}})^{-1}(U) = f^{-1}(U) \cap A_{m+1}$, which is open in A_{m+1}. This means that there is an open set $V \subset X$ such that $V \cap A_{m+1} = f^{-1}(U) \cap A_{m+1}$. Define $W = V \cap A^{\circ}_{m+1}$, which is open in X. We know that $p \in A_m \subset A^{\circ}_{m+1}$, and $p \in f^{-1}(U) \cap A_{m+1} = V \cap A_{m+1}$ implies that $p \in V$, so we can conclude $p \in W$. Finally, note that $W = V \cap A^{\circ}_{m+1} \subset V \cap A_{m+1} = f^{-1}(U) \cap A_{m+1} \subset f^{-1}(U)$, so we conclude $W \subset f^{-1}(U)$. Thus, W is an open set containing p, and is contained in $f^{-1}(U)$. We conclude that $f^{-1}(U)$ is open, for any open $U \subset Y$, and therefore f is continuous.

3 - Compactness and Connectedness

Exercise 3 (*page 47*)

Let I be the (compact) intervall, and let $S \subset I$ be infinite. Suppose S has no limit points in I. Then there is an open U_x for each $x \in I$ such that $x \in U_x$ and $(U_x - \{x\}) \cap S = \emptyset$. Since $\{U_X | x \in I\}$ forms an open cover of I, Heine-Borel tells us that there are $x_1, \ldots, x_n$ such that $I \subset U_{x_1} \cup \cdots \cup U_{x_n}$. Each U_{x_i} contains at most one point from S and therefore S must be finite – a contradiction.

Exercise 6 (*page 50*)

Let $X = [0,1]$ as a subspace of $\mathbb{E}$. Let $Y = [0,1]$ as a subspace of $\mathbb{R}$ with the co-finite topology. Let $f : X \to Y$ with $f(x) = x$, for all $x \in X$. It is clear that f is 1-1 and onto, and since all co-finite sets are also open, f is continuous. On the other hand $f([0,1)) = [0,1)$ which is open in X but not in Y. Thus f^{-1} is not continuous. ∎

Exercise 7 (*page 50*)

We begin by defining

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$$
\begin{aligned}
&x_0 = 0 \\
&x_n = 1 + \tfrac{1}{2} + \dots \tfrac{1}{n} \\
&r_n = \tfrac{1}{n+1} \\
&U = \mathbb{E}^2 - \{(x,0) : x \geq 0\} \\
&B_n = B(x_n, r_n), n \geq 0
\end{aligned}
$$

We claim that $\{U, B_1, \dots\}$ constitutes an open cover of $\mathbb{E}^2$. (To see that it is really a cover, first note that (x_n) diverges. Therefore, for any $a \geq 0$, there is a largest N such that $x_N \leq a$. This implies that $|x_N - a| < r_n$, i.e. $(a, 0) \in B_N$. In other words, any point in U^c is contained in some B_n, so we have a cover. It should be clear that the selected sets are open.)

We will now see that this open cover shows that Lebesgue's lemma doesn't hold for $\mathbb{E}^2$. Pick some $\delta > 0$, and pick N large enough that $r_N < \delta$. Then the open ball $B((x_N, 0), \delta)$ is not contained U nor in any B_n. ▮

Exercise 11 (*page 50*)

Let $\mathfrak{B} = \{(-\infty, a) : a \in \mathbb{R}\} \cup \{\emptyset\} \cup \{\mathbb{R}\}$. By theorem 2.5, this forms the basis for some topology on $\mathbb{R}$. In this topology, $[0, 1]$ is compact, but it's closure is $[0, \infty)$, which is not compact. ▮

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