1 - Functions on Euclidean Space

1 - Norm and inner product

Exercise 7 (*page 4*)

(a) Suppose T preserves inner product. Then $|Tx| = \sqrt{\langle Tx, Tx\rangle} = \sqrt{\langle x, x\rangle} = |x|$. Thus T is norm preserving. Now suppose T preserves norm. Then by theorem 1-2(5), we have $\langle Tx, Ty\rangle = \frac{|Tx+Ty|^2-|Tx-Ty|^2}{4} = \frac{|T(x+y)|^2-|T(x-y)|^2}{4} = \frac{|x+y|^2-|x-y|^2}{4} = \langle x, y\rangle$, and so T preserves inner product. **(b)** If T was not 1-to-1, then there would exist x, y with $x \neq y$ (i.e. $|x-y| > 0$), yet with $Tx = Ty$ (i.e. $|Tx - Ty| = 0$. But by hypothesis $|Tx - Ty| = |x - y| = 0$, so T must have an inverse, which is clearly also norm preserving (and therefore it also preserves the inner product).

2 - Differentiation

5 - Inverse functions

Exercise 37 (*page 39*)

(a) Define $g : \mathbb{R}^2 \to \mathbb{R}^2$ as $g(x, y) = (f(x, y), y)$. The differential is then

$$g'(x, y) = \begin{pmatrix} D_1 f(x, y) & D_2 f(x, y) \\ 0 & 1 \end{pmatrix}$$

and so $\det g'(x, y) = D_1 f(x, y)$. Now, if f is $1-1$, then there must exist some point (a, b) at which $\det g'(a, b) \neq 0$.

4 - Integration on Chains

1 - Algebraic Preliminaries

Exercise 4 (*page 85*)

Since $\dim(\Lambda^n(\mathbb{R})) = 1$, and since $\omega \in \Lambda^n(V)$ implies that $f^*\omega \in \Lambda^n(\mathbb{R}^n)$, we know that $f^*\omega = c \det$ for some $c \in \mathbb{R}$. To find out c, we evaluate this equation at $(e_1, \ldots, e_n)$:

$$f^*\omega(e_1, \ldots, e_n) = c \det \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix} = c$$

or

$$\omega(f(e_1), \ldots, f(e_n)) = c$$

Applying Theorem 4-6, we get

$$det(a_{ij}) \cdot \omega(v_1, \ldots, v_n) = c$$

where $\{v_i\}_{i=1}^n$ is a basis for v and the a_{ij} are defined by $f(e_i) = \sum_{j=1}^n a_{ij} v_j$. By hypothesis, $[f(e_1), \ldots, f(e_n)] = \mu$ so that $\det(a_{ij}) = 1$. Also, since ω is the volume element w.r.t. μ and T, we get $\omega(v_1, \ldots, v_n) = 1$, so we conclude $c = 1$. Thus $f^*\omega = \det$ follows.

4 - The Fundamental Theorem of Calculus

Exercise 29 (*page 105*)

To show *existence*, define $\lambda = \int \omega = \int_0^1 f dx$, and $g(x) = \int_0^x f dx - \lambda x$. By Theorem 4-7, we calculate $dg = Dg \cdot dx$. $Dg = f - \lambda$, so $dg = \omega - \lambda dx$ follows. Also note that $g(0) = \int_0^0 f dx - \lambda \cdot 0 = 0$, and $g(1) = \int_0^1 f dx - \lambda = \int_0^1 f dx - \int_0^1 f dx = 0$, so $g(0) = g(1) = 0$.

To show *uniqueness*, suppose that $\omega - \lambda dx = dg$, where $g(0) = g(1)$. Integrating, we get

$$\int_0^1 \omega - \lambda = \int_0^1 dg$$

By Stoke's theorem, we have $\int_0^1 dg = g(1) - g(0) = 0$, so $\lambda = \int_0^1 \omega = \int_0^1 f dx$. So the λ is completely determined by f, and is thus *unique*.

Exercise 34 (*page 108*)

(a) Following the definitions, we have

$$\partial C_{F,G} = -(C_{F,G})_{(1,0)} + (C_{F,G})_{(1,1)} + (C_{F,G})_{(2,0)} - (C_{F,G})_{(2,1)} - (C_{F,G})_{(3,0)} + (C_{F,G})_{(3,1)}$$

where

$$(C_{F,G})_{(1,0)}(u,v) = C_{F,G}(0,u,v) = F(0,u) - G(0,v) = C_{F_0,G_0}(u,v)$$
$$(C_{F,G})_{(1,1)}(u,v) = C_{F,G}(1,u,v) = F(1,u) - G(1,v) = C_{F_1,G_1}(u,v)$$
$$(C_{F,G})_{(2,0)}(s,v) = C_{F,G}(s,0,v) = F(s,0) - G(s,v)$$
$$(C_{F,G})_{(2,1)}(s,v) = C_{F,G}(s,1,v) = F(s,1) - G(s,v)$$
$$(C_{F,G})_{(3,0)}(s,u) = C_{F,G}(s,u,0) = F(s,u) - G(s,0)$$
$$(C_{F,G})_{(3,1)}(s,v) = C_{F,G}(s,u,1) = F(s,u) - G(s,1)$$

Since each F_s is closed, i.e. $F(s,0) = F(s,1)$, for all s, we see that $(C_{F,G})_{(2,0)} = (C_{F,G})_{(2,1)}$.

Similarly, since each G_s is closed, $(C_{F,G})_{(3,0)} = (C_{F,G})_{(3,1)}$. Thus, the only surviving terms of $\partial C_{F,G}$ are those due to $(C_{F,G})_{(1,0)}$ and $(C_{F,G})_{(1,1)}$. So $\partial C_{F,G} = C_{F_1,G_1} - C_{F_0,G_0}$ follows.

(b) If $d\omega = 0$ then by (a) and Theorem 4-13 (Stoke's theorem), we get

$$0 = \int_{C_{F,G}} d\omega = \int_{\partial C_{F,G}} \omega = \int_{C_{F_1,G_1}} \omega - \int_{C_{F_0,G_0}} \omega$$

so we conclude $\int_{C_{F_1,G_1}} \omega = \int_{C_{F_0,G_0}} \omega$.

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