We begin by defining

$$\begin{aligned}
&x_0 = 0 \\
&x_n = 1 + \tfrac{1}{2} + \dots \tfrac{1}{n} \\
&r_n = \tfrac{1}{n+1} \\
&U = \mathbb{E}^2 - \{(x, 0) : x \geq 0\} \\
&B_n = B(x_n, r_n), n \geq 0
\end{aligned}$$

We claim that $\{U, B_1, \dots\}$ constitutes an open cover of $\mathbb{E}^2$. (To see that it is really a cover, first note that (x_n) diverges. Therefore, for any $a \geq 0$, there is a largest N such that $x_N \leq a$. This implies that $|x_N - a| < r_n$, i.e. $(a, 0) \in B_N$. In other words, any point in U^c is contained in some B_n, so we have a cover. It should be clear that the selected sets are open.)

We will now see that this open cover shows that Lebesgue's lemma doesn't hold for $\mathbb{E}^2$. Pick some $\delta > 0$, and pick N large enough that $r_N < \delta$. Then the open ball $B((x_N, 0), \delta)$ is not contained U nor in any B_n. ∎

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